Erich Harasko *)

Spoke Diagrams -
A New View of the Knots and Links.

A new representation of the knots and links is introduced, called ‘spoke diagrams’, which consist of disjoint circles in the plane (Seifert-circles), connected in the plane without crossings by so called ‘spokes’.

For these spoke diagrams are deduced the changes by Reidemeister moves. Since they are dependent on orientation, we get for the R-move 2 always 2 cases:  (+R2a), and  (+R2b). Similarly we get also 2 cases for R-move 3.

After demonstration of different examples it is proved, that the arbitrary consecutive application of (+R2b) between always 2 circles at an arbitrary given spoke diagram always leads to a centred diagram (thus replacing the Vogel-algorithm). The transfer of this simple method into a treatment of the customary diagrams is not possible, because in these cannot be distinguished, whether the 2 ropes of the Reidemeister move R2 belong to 2 different (Seifert-) circles or to only 1.

The advantageous use of the centred spoke diagrams at the recurrence relation $AX(+) + BX(-) + CX(0) + D = 0$ for purpose of determining knot-invariant polynomials is demonstrated. For its finite recursion process is given a proof by the so called twin spokes theorem, which is proved before.

As appendix are given centred spoke diagrams of the prime- knots and links until 8 spokes, represented by a simple numerical code according to the algebraic word of their view as closed braids, which is specifically suited for computerizing.

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*) Dipl.-Ing. of the Technical University at Vienna 1961.

1. Representation of Knots and Links by Spoke Diagrams.

Let be given a regular projection of a knot *K*. By one pass through *K* (in an arbitrary direction) we may associate 2 pointers to each crossing, the starting-point of which is always assumed before the crossing, and the end-point is assumed after the crossing (s. Fig. 1). Afterwards we may turn the 2 pointers (more exactly: the line pieces, which they do represent) around the crossing-point from each other, until they fall into one line in projection, but in opposite directions (s. Fig. 2 and 3).

Instead of a crossing-point we then always get a crossing piece of line, which is called a ‘spoke’ (s. Fig. 3). By one pass through *K* each spoke is in the projection passed exactly 2 times, namely in opposite directions. All the other lines of the projection are passed through just one time.
During one pass through *K* the pass through the projection takes the following way: if we come on a normal line to the end-point S_1 of a spoke (s. Fig. 3) the pass must be continued through the spoke until point S_2 , from which it is continued according to the following

<u>Rule of running through:</u>

> If the spoke was reached by turn off right, the pass after the spoke is continued by turn off left. On the other hand – if the spoke was reached by turn off left, the pass after the spoke is continued by turn off right.

Now we ask for the 1-time passed projection-lines p_i . Let us add the endpoints of each spoke to the p_i ; thus the separated lines p_i , lying at the end of each spoke will always form there a connected line k_j .The line-pieces p_i have no crossings and because of the degree 2 of the points, where they were connected, the lines k_j must form disjoint circles (Seifert-circles [3]).

If on a circle-piece besides S_1 (s. Fig. 3) the circle is run through towards S_1 , it must be continued through the spoke; the opposite run through the spoke must then be continued in S_1 from S_1 off through the other circle-piece. The run through the pieces of the circle therefore always defines a certain orientation of the circle.

In the following is still shown that the 2 endpoints of a spoke always belong to 2 different circles.
Let’s assume the endpoints S_1 , S_2 of a spoke would lie at the same circle. In this case the ‘rule of running through’ would define a *different* orientation of the circle in S_1 and S_2 (s. Fig. 4) – in contradiction to the result we have just found above. ☐

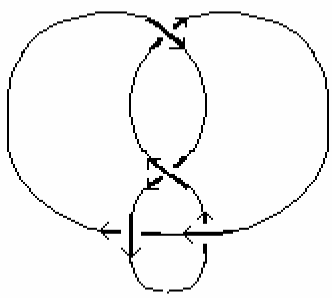


Fig. 1

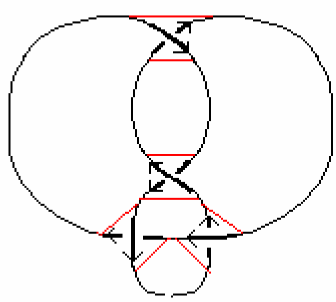


Fig. 2

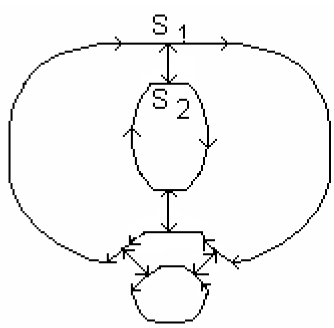


Fig. 3

Therefore follows:

<u>Theorem 1</u>:

> A spoke is always situated between 2 different circles and for a knot-projection with $n > 0$ crossings we get at least 2 and maximally $(n+1)$ disjoint oriented circles in the plane, which are connected by n spokes.
> If the orientation in one of the circles is chosen, the orientation of all the other circles is defined by the 'rule of running through'.

Finally we assign to each spoke a sign to get a one-to-one representation of the spoke and the pass through the knot, which will be reached by the following

<u>Rule for the sign:</u>

> A spoke is assigned with '+' , if the pointer of the oriented piece of knot that lies *below*, had to be turned around anticlockwise, in order to fall in one line (but with opposite direction) with the one that lies above.. If it had to be turned around clockwise, the spoke is assigned with '-' .

It may be noticed, that this rule is independent from the chosen direction of the pass through the knot.

Fig. 5 finally shows the complete spoke diagram of the knot in Fig. 1.

On the other hand, if now the 2 pointers are again turned from each other and the pointer that lies below is determined in such a way, that the 'rule for the sign' is fulfilled, we again get (after taking away the small circle-pieces, which lie in the turning angle) the regular knot-projection (s. Fig. 6).

Links differ from knots by crossings, whose components belong to *different* knots of the link. In case of a knot the spoke diagram is independent from the chosen orientation of the pass through (*both* pointers of each crossing do then have the inverse direction, which leads to the same diagram), but in case of a link the spoke diagram generally depends from the chosen orientations in its components, -we thus get quite different spoke diagrams (in the same way as also the Jones-polynomial does give different results for these cases).

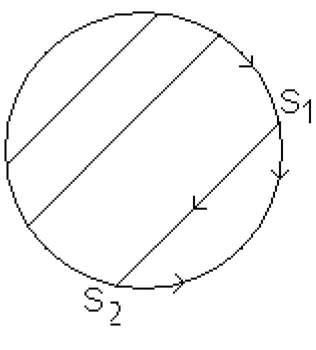


Fig. 4

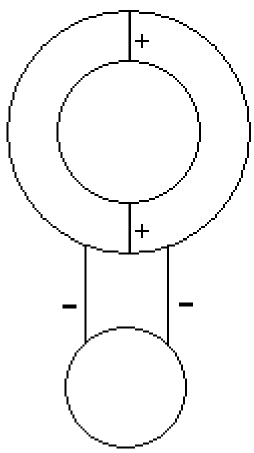


Fig. 5

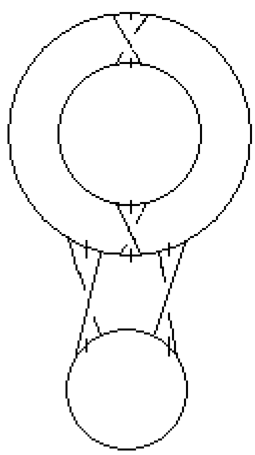


Fig. 6

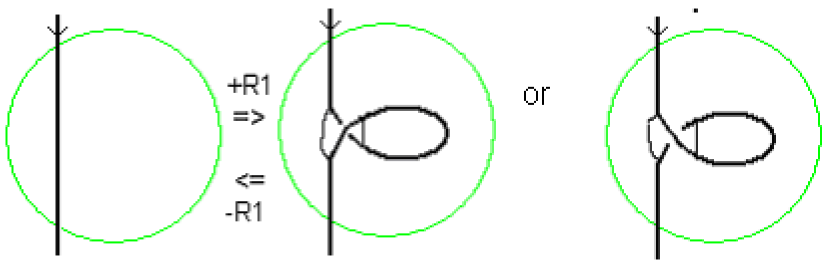
The spoke diagram of a link differs from the spoke diagram of a knot only by the fact, that during one pass-through the starting-point is reached before all spokes were run through in both directions. In this case an other pass-through (that of an other knot of the link) is started at any of those spokes. In this way must be continued until all spokes are run through in both directions, that means, that we had a run through all the components of the link.

2. The Change of the Spoke Diagrams by the Reidemeister Moves.

To recognize these changes, we only need to consider the part of the knot, which Is changed by the Reidemeister moves. In the following figures this part is always separated from the rest, which is of no interest, by a thin circle. Within the circle the knot is drawn in its complete structure, but from the outside part are only drawn short pieces of the lines with their orientation, which are leading into the circle.

At first is always shown the part of the knot before and after the Reidemeister move, represented in regular projection, and afterwards the same, represented by spoke diagrams.

<u>R1</u>:



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Spoke diagrams:

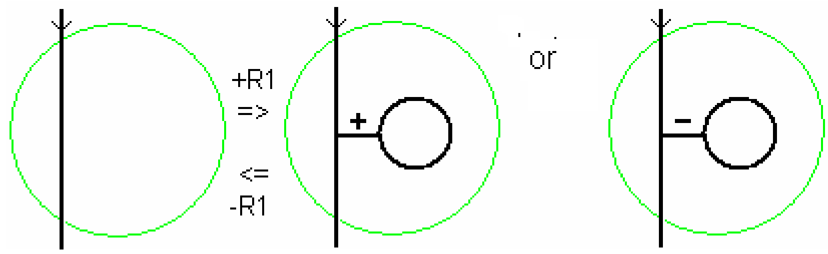
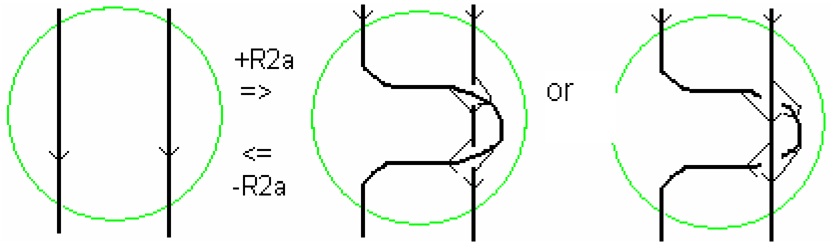


Fig. 7 The sign of the crossing is independent from the orientation, the result is therefore the same for both orientations of the knot.

<u>R2a</u>:



Spoke diagrams:

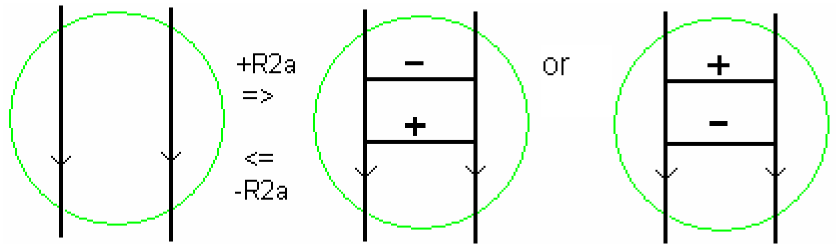
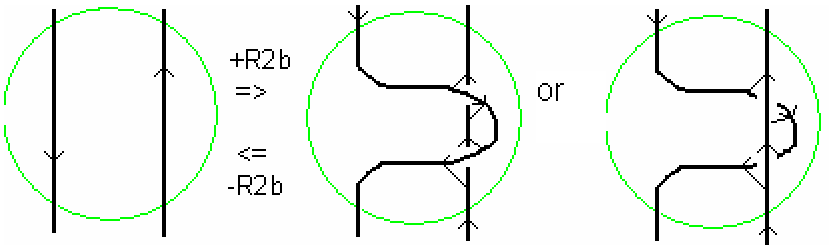


Fig. 8

Now must still be considered the second possible case, the one of inverse orientation in one of the two lines.

<u>R2b</u>:



Spoke diagrams:

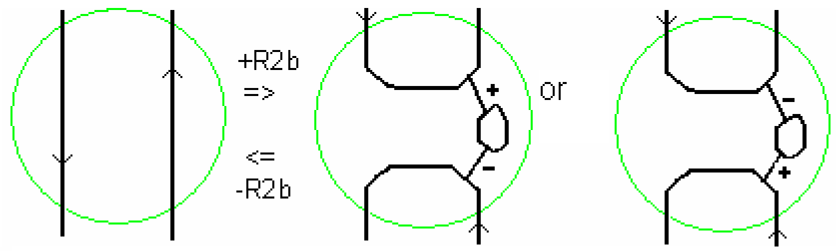


Fig. 9

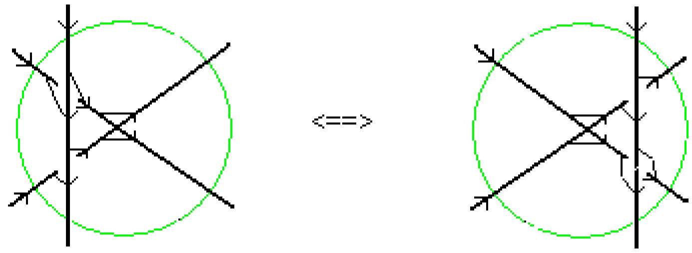
In studying the 3rd Reidemeister move, we will not draw both cases – where either the shifted line is lying *above* the crossing or *below* it – but only the first one. In the other case the two signs are inverted, which will be represented by the following

<u>Rule for the marking of coupled signs</u>:

For example (⊕, ⊖); the ringing means, that either the signs may be those as declared or *both* the inverted ones.

A single crossing with a sign, which may be chosen arbitrary (independent from others) is represented by both signs together (+ -).

R3/ Case 1:



Spoke diagrams:

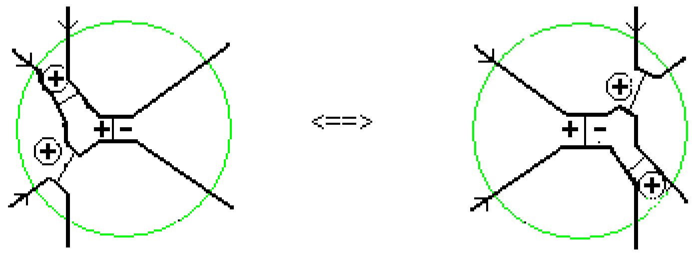
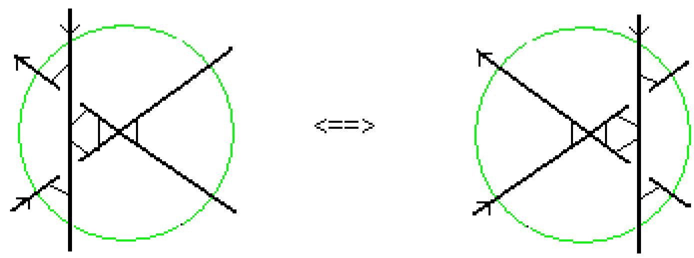


Fig. 10

Because the spoke diagrams are also dependent from the orientation of the lines leading into the circle, we will get 8 different cases.

But since the inverted orientation of all 3 lines leads to the figure on the right side, which differs from that on the left only by a turn of 180°, we only must study 4 cases, in which never 2 have inverted orientations to each other.

R3/ Case 2:



Spoke diagrams:

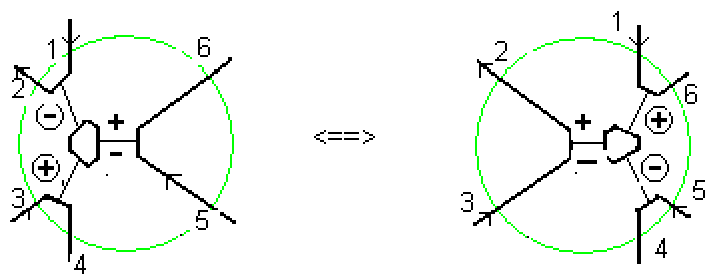
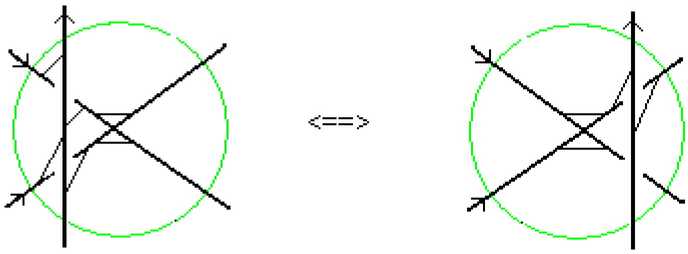


Fig. 11

The circle-pieces (1, 3), (3, 4), (5, 6) of the former spoke diagram vanish and the new circle-pieces (2, 3), (4, 5), (6, 1) are added. The inner circle with 3 spokes is turned by 180°.

R3/Case 3:



Spoke diagrams:

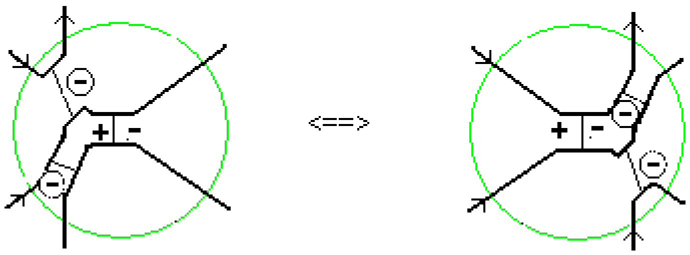
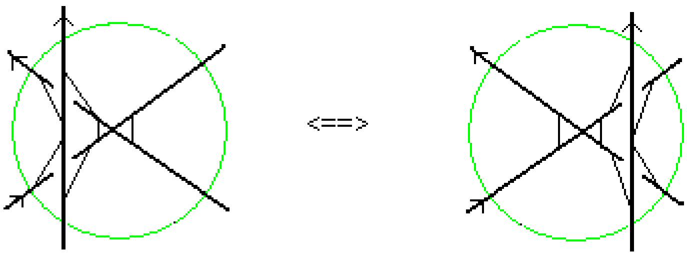


Fig. 12

Case 3 leads (with respect to the rule for coupled signs) to case 1.

R3/ Case 4:



Spoke diagrams:

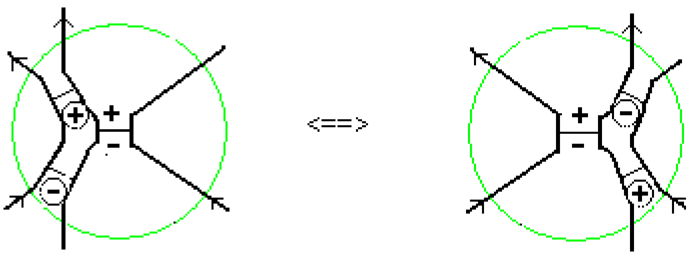
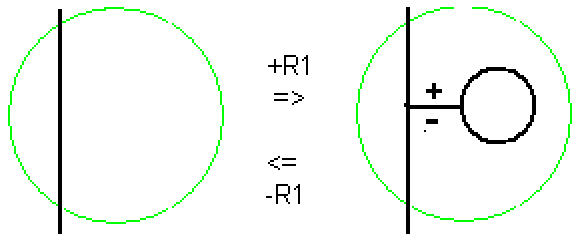


Fig. 13

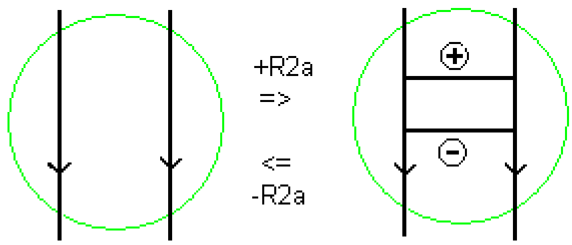
Case 4 is represented by case 1. Independent from the spoke, whose sign may be arbitrary chosen, there always exists a configuration in case 1, by which it is represented.

Summary of the changes by the Reidemeister moves

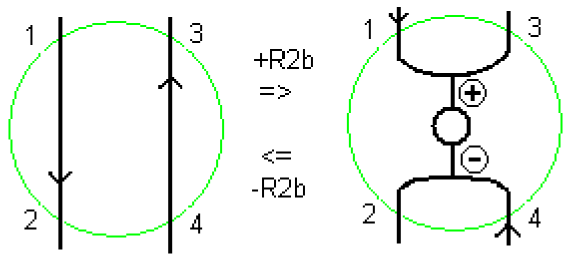
R1:



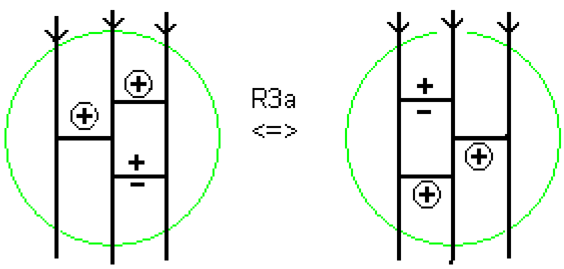
R2a:



R2b:



R3a:



R3b:

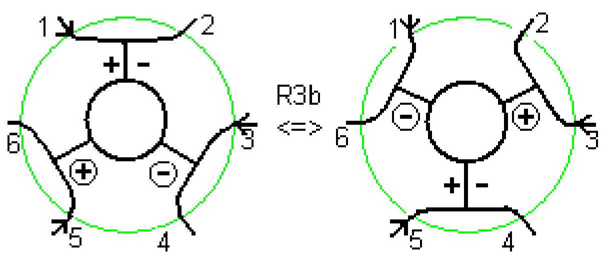


Fig. 14

3. Special Equivalent Changes for Spoke Diagrams.

The following knot-invariant changes for spoke diagrams do whether change the number of crossings nor the number of circles. The basis of this is - the sign of a spoke is independent from the direction in which the knot is considered.. Therefore the whole knot or parts of it may be turned around without change of the signs.

Equivalent changes:

a) Turning of the whole knot *k*: in this case the spoke connections in each circle of the turned knot W(*k*) are situated in the opposite way round.

b) Turn of a knot *k*, which surrounds the rest of the knot (s. Fig.15). The part of the knot lying in the inner of *k* is afterwards lying besides *k*. The spoke connexions 1, 2, 3 in *k* are afterwards situated in opposite way.

For concentric circles is valid:

c) Turn upside down of a concentric spoke diagram *k*: By always applying change (b) we do finally reach the turned upside down spoke diagram P(*k*), where the most outside lying circle became the most inside lying one and inverse. All connexions of the spokes are then the other way round.

d) Overthrow S(*k*) of a concentric spoke diagram *k*: results by applying P(*k*) and afterwards W(*k*) to *k*. The most outside lying circle will become the most inside one and inverse, but all the connexions are situated the same way round as before.

In Fig. 16 as an example to a concentric knot *k* are shown the equivalent changes W(*k*), P(*k*) and S(*k*). Generally is valid

Theorem 2:

A knot *K*, represented by a concentric spoke diagram *k*, is equivalent represented by the turned diagram W(*k*), turned upside down diagram P(*k*), and the overthrown diagram S(*k*). It is valid:

$S(k) = P(W(k)) = W(P(k))$ (3.1)
$P(k) = W(S(k)) = S(W(k))$ (3.2)
$W(k) = S(P(k)).= P(S(k)).$ (3.3)

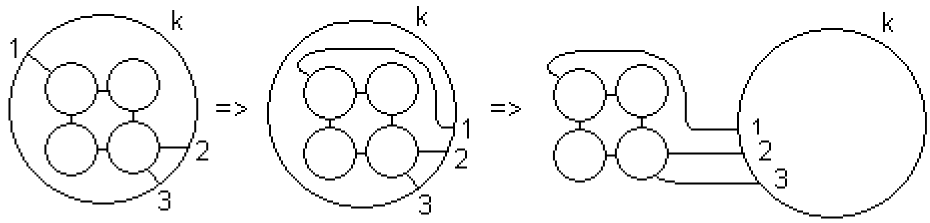


Fig. 15

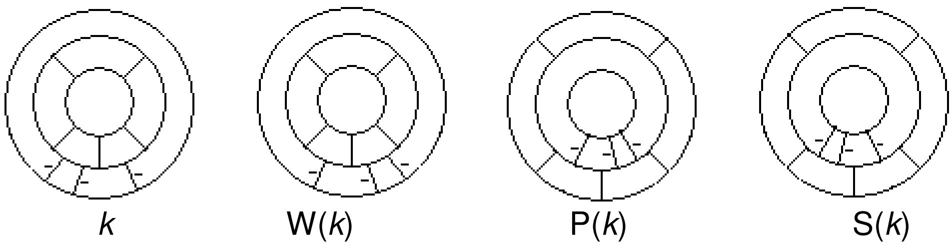


Fig. 16

4. Orientation of Circles in Spoke Diagrams

If in one of the circles of the spoke diagram of a knot an orientation is chosen, the orientation of all the other circles of this spoke diagram are defined by the 'rule of running through'.

Two circles may have two different positions to each other: one may lie in the inside of the other or they may lie besides each other. Because of the 'rule of running through' we do get

Theorem 3

> Two circles lying besides each other, which are connected by spokes, have :an opposite orientation to each other.
> Two circles,, from which one is lying inside the other and which are connected by spokes, do always have the same orientation.

In one pass through the spoke diagram of a knot, all circle-pieces lying between the spokes, will be passed exactly one time in direction of the orientation of the circle and each spoke exactly one time in each of its directions.

In a spoke diagram of a link of knots one pass through the whole spoke diagram will not be reached before all of its knots are passed through. Corresponding to the definition of spoke diagrams in paragraph 1, this will be reached if all its spokes are run through in each of its directions.. If this condition is not yet reached, one can start a pass through of an other knot at one of the named spokes.

On the other hand is valid

Theorem 4:

> Each set of disjoint circles in the plane, being connected in the plane in such a way, that theorem 3 is valid, represents the spoke diagram of a knot respectively a link of knots.

Proof: If theorem 3 is fulfilled, then the 'rule of running through' is valid – and in this case the spoke diagram can be transformed according to the figures 5 and 6 into a regular projection of a knot or link. □

5. Examples for the Application of Equivalent Changes.

a) Identification of the knot in Fig. 17.1 as unknot.

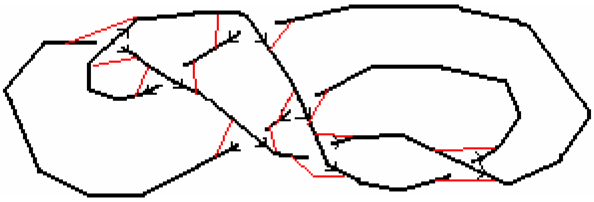


Fig. 17.1

In Fig. 17.2 left is shown the representation of the knot as a knot diagram. By turning up the right concentric system of circles over the left one – according to the equivalent changes, which are valid for spoke diagrams (s. paragraph 3) – we get a single concentric spoke diagram (s. Fig 17.2 right).

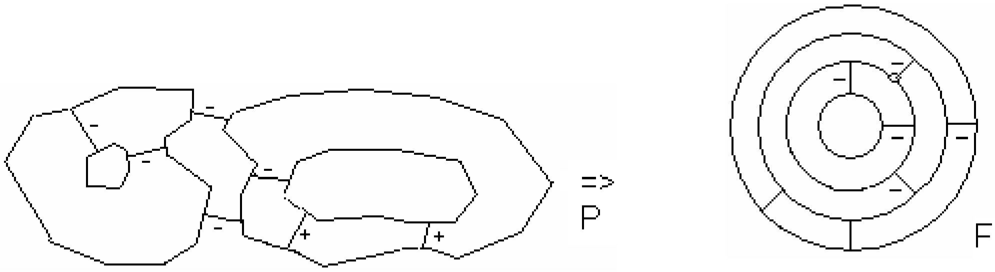


Fig. 17.2

In order not moving aimless in space of Reidemeister, R3a-moves at a concentric spoke diagram will be further on made in such a way, that the number of spokes in the outer ring will always be increased. The place, at which the Reidemeister move concerns, will always be marked by a small circle.

Fig.17.3 shows the equivalent changes, which are leading to the unknot. □

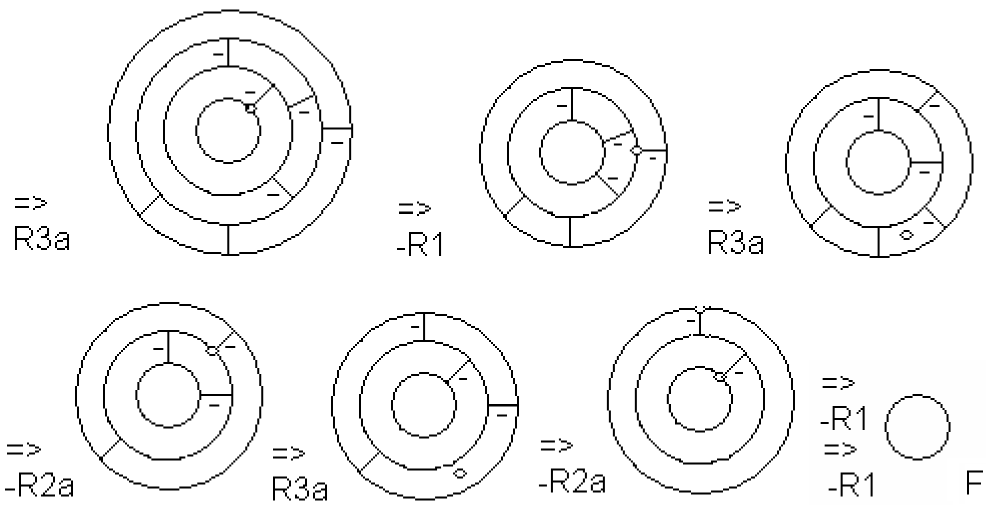


Fig. 17.3

b) Equivalence of the links V1² and V2² of Fig. 18.

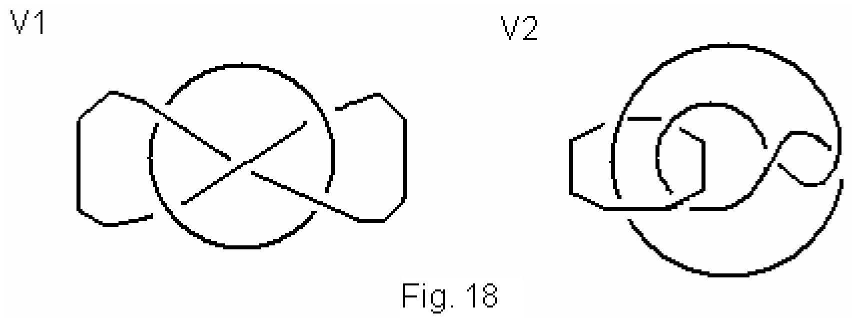


Fig. 18

Depending on the orientations of the knots of the link one generally gets different diagrams.

For the equivalence of two links V1 and V2 is necessary and sufficient , that the set of all possible orientations of the knots of V1 leads to the same result as the set of all possible orientations of the knots of V2.

In the case of a link of 2 knots we get only 2 possible orientations: a and b, - (the inverse orientation in each of the knots does always lead to the same result). In the following we therefore find out V1(a) and V1(b), further on V2(a) and V2(b) and show, that {V1(a), V1(b)} = {V2(a). V2(b)] (s. Fig. 19.1 and 19.2).

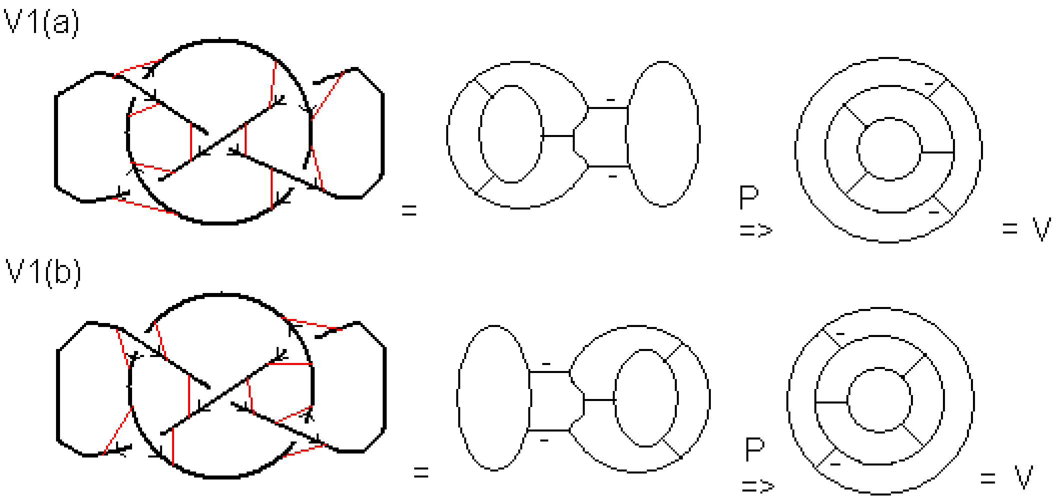


Fig. 19.1

'*m**R3a' means in the following an *m*-times made R3a-move with always the same (marked) spoke.

Coloured circles only should bring more overview. (+R2b)-moves are represented by 2 green lines, leading between 2 circles, and connected over a red circle.

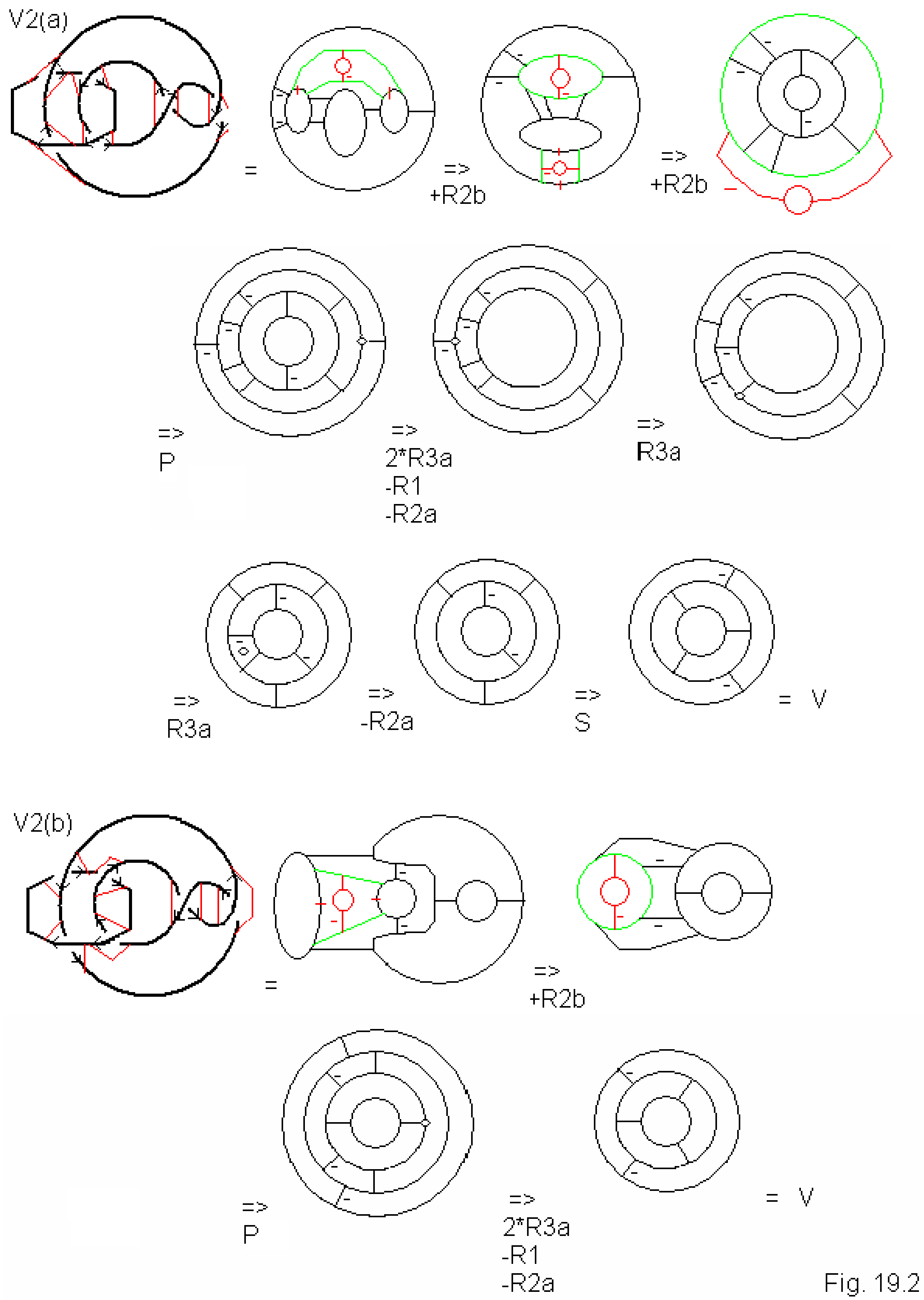


Fig. 19.2

The result is {V1(a), V1(b)} = {V, V} = {V2(a), V2(b)}. □

c) Equivalence of the pair of 'Perko'-knots (Fig. 20 and 21).

Perko2:

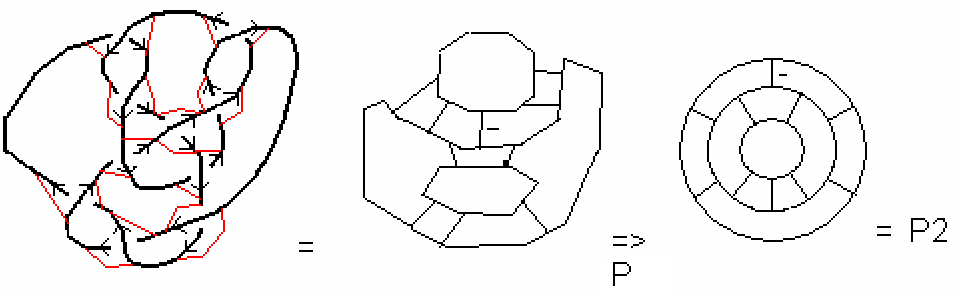


Fig. 20

Perko1:

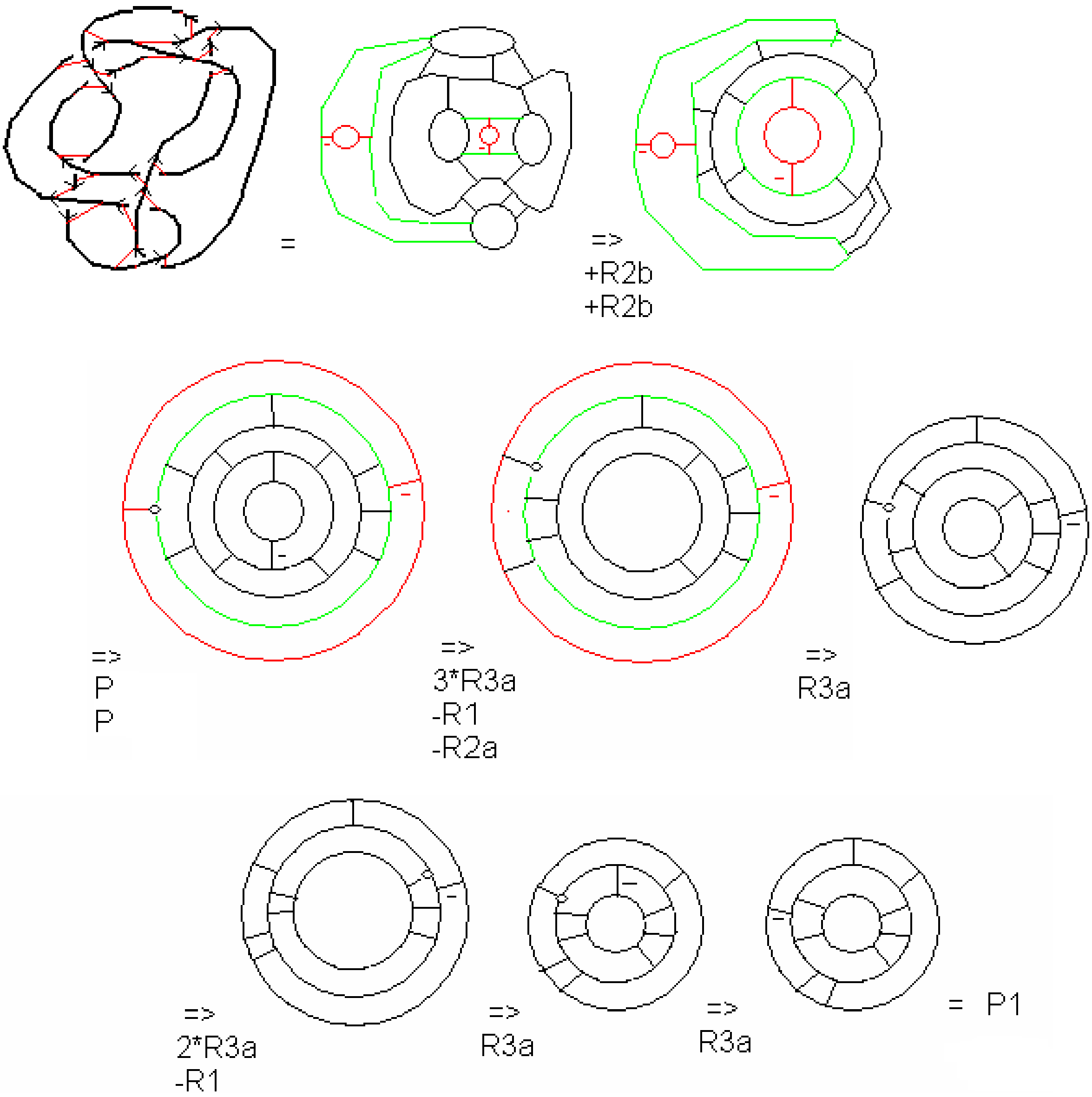


Fig. 21

The concentric spoke diagrams P1 and P2 can be immediately recognized as equivalent spoke diagrams. □

6. The (+R2b)-Method of Centring.

In all the examples of section 5 could be seen, that an arbitrary consecutive application of the Reidemeister move (+R2b) between always 2 circles led to a centric spoke diagram, respectively a bi-centric one, which by turning one upside down the other can always be transformed into a centric one. In the following is proved, that this is valid generally, thus replacing the Vogel-algorithm [3].

In fact – the bi-centric and the concentric diagram are the only ones, at which no further (+R2b)-move between any 2 circles is possible.

Fig. 22.1 and 22.2 show the 2 (according to Fig. 14) possible cases of applications of (+R2b) at 2 different circles K1, K2.

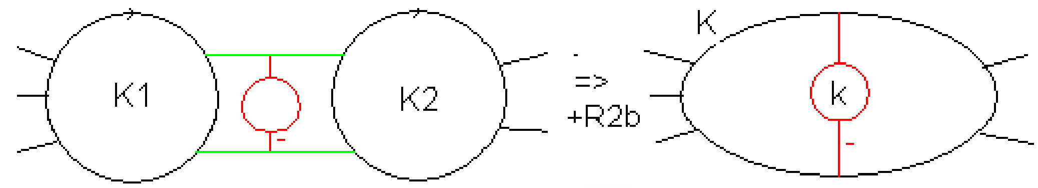


Fig. 22.1

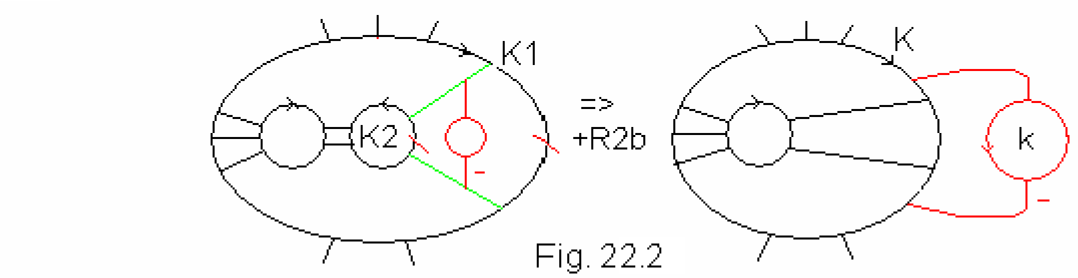


Fig. 22.2

In Fig. 22.1 K1 and K2 are lying besides each other and have the same orientation (R or L). By the execution of (+R2b) K1 and K2 change into the circles K and k, where k is lying inside of K, both having again the same orientation as K1 and K2. This case is called <u>'concentration of K1 and K2'</u>.

In Fig. 22.2 K1 and K2 are lying one inside the other and have different orientations (R and L or L and R). By the execution of (+R2b) K1 and K2 change into the circles K and k, lying besides each other, where both circles again have different orientations, K as K1 and k as K2. This case is called <u>'outside-transfer of K2'</u>.

Because of the above described facts is valid:

<u>Theorem 5</u>:

The number of R-circles as well as the number of L-circles is not changed by the execution of a Reidemeister move (+R2b) between 2 different circles.

<u>Defiinitions:</u>

$r(R_i)$ is the number of right-oriented circles (R-circles), which surround the R-circle R_i.
$l(R_i)$ is the number of left-oriented circles (L-circles), which surround R_i.
$l(L_i)$ is the number of L-circles, which surround the L-circle L_i.
$r(L_i)$ is the number of R-circles, which surround the L-circle . L_i.

r(R) is the sum of the number of R-circles, which surround the different R-circles.
l(R) is the sum of the number of L-circles, which surround the different R-circles.
l(L) is the sum of the number of L-circles, which surround the different L-circles.
r(L) is the sum of the number of R-circles, which surround the different L-circles.

Now can be defined the degree of concentration by the following 'concentration-number' *z:*

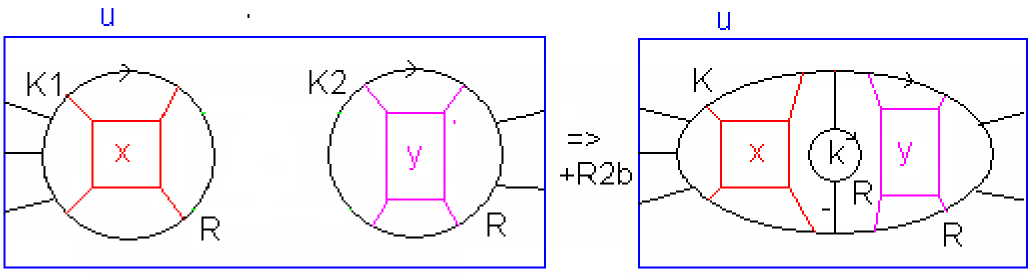
$$z = [r(\mathrm{R}) - l(\mathrm{R})] + [l(\mathrm{L}) - r(\mathrm{L})] \tag{6.1}$$

<u>Theorem 6:</u>

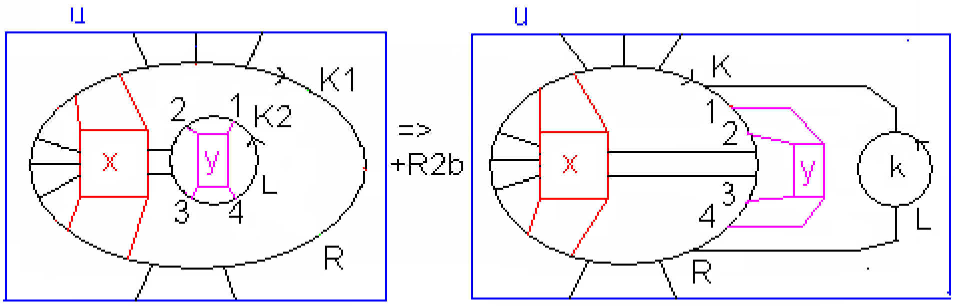
An arbitrary (+R2b)-move always enlarges the concentration-number *z* by exactly 1.

Proof:

Fig. 23.1 shows the general case of a concentration and Fig. 23.2 the general case of an outside transfer. The difference to Fig. 22.1 and Fig. 22.2 are the fields x, y and u, which may contain an arbitrary number of R-circles (R_x, R_y, R_u) and L-circles (L_x. L_y. L_u).



(a) Fig. 23.1 (b)



(a) Fig. 23.2 (b)

Let's now consider the change of *z* by a (+R2b)-move (s. Fig. 23.1 and 23.2): $\triangle z = z_b - z_a$.

From Fig. 23.1 can be seen, that the number of surrounding R-circles as well as L-circles is for circles of field x as well as circles of field y in Fig. (b) the same as in Fig. (a). That means, that they need not to be considered for $\triangle z$.

From Fig. 23.2 can be seen, that the number of surrounding R-circles as well as that one of surrounding L-circles for a circle in x is the same, but a circle in y of (b) is surrounded by 1 L-circle (K2) less and by 1 R-circle (K1) less than in (a). According to (6.1) we get for each R-circle as well as L-circle of y: $\triangle z = [(-1) - (-1)] = 0$.

For $\triangle z$ we therefore only need always to consider the circles K1,K2 and K, k.

At *concentration* (s. Fig. 23.1) is valid according to (6.1):

$\triangle z = [(r(K) + r(k)) – (l(K) + l(k))] – [(r(K1) + r(K2)) - (l(K1) + l(K2))] =$

$= [r(K) - r(K1)] + [r(k) - r(K2)] + [l(K1) - l(K)] + [l(K2) - l(k)].$

Under consideration, that circles of u are always surrounding K1, K2 in the same way as K, k, we get for the last result immediately from Fig. 23.1 :

$= [0] + [(r(K2)+1) - r(K2)] + [0] + [0] =$

$= +1.$ (6.2)

At *Outside transfer* (s. Fig. 23.2) is valid according to (6.1):

$\triangle z = [r(K) - l(K)] + [l(k) - r(k)] – [r(K1) - l(K1)] – [l(K2) - r(K2] =$

$= [r(K) - r(K1)] + [l(K1) - l(K)] + [l(k) - l(K2)] + [r(K2) - r(k)].$

Under consideration, that circles of u are always surrounding K1, K2 in the same way as K, k, we get for the last result immediately from Fig. 23.2 :

$= [0] + [0] + [0] + [(r(k) +1) - r(k)] =$

$= +1.$ (6.3)

□

<u>Theorem 7</u>:

If z is the concentration number of a given spoke diagram d, and n_R, n_L its number of R- respectively L-circles, then is the necessary and also sufficient number of (+R2b)-moves, which transform d into a bi-centric spoke diagram, equal to $\frac{1}{2}[(n_R-1)n_R + (n_L-1)n_L] – z$.

Proof:

The bi-centric spoke diagram consists of 2 concentric systems, connected by spokes and therefore with different orientations. That means, that at this, as well as at a concentric spoke diagram (case $n_L = 0$), no further (+R2b)-move is applicable (compare Fig. 14). Note, that this is not the case at any other spoke diagram.

Because of theorem 5 the bi-centric system has the concentration number

$$z_c = (1+2+\ldots+(n_R -1)) + (1+2+ \ldots+(n_L-1) = \tfrac{1}{2}[(n_R -1)n_R +(n_L -1)n_L]. \quad (6.4)$$

As long as there is not reached a bi-centric or concentric diagram, it must be possible to apply a (+R2b)-move. Then – after theorem 6 - the concentration number is enlarged by 1, in this way repeating until we get z_c and have reached a bi-centric respectively concentric diagram. Thus it is not only proved, that we have a finite process, but also predicted the number of steps, which sufficiently lead to the result.

□

From the bi-centric spoke diagram can simple be reached the intended concentric one by turning any of the 2 concentric parts upside down the other (s. equivalent change P in par.3).

Dependent on the places, at which the (+R2b)-connections are put on at the circle, and dependent on the always alternative assignment of the +/- pair of spoke-signs, we get a lot of different concentric diagrams for one certain knot (oriented link). Each of them is reached from any other one of them by a sequence of R-moves.

On the other hand, as immediate consequence of theorem 4 is valid

Theorem 8:

By any concentric spoke diagram a knot respectively an oriented link of knots is well defined.

In the following are shown special applications of the (+R2b)-method.

a) Concentration of 2 systems of concentric circles (Fig.24 a.b.c and 25 a,b,c):

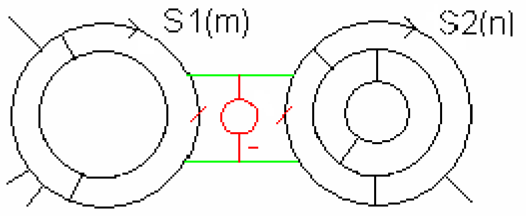


Fig. 24 a

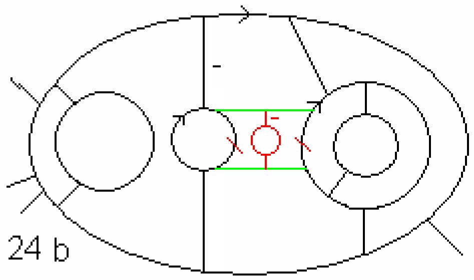


Fig. 24 b

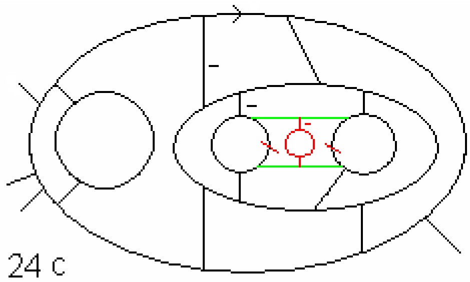


Fig. 24 c

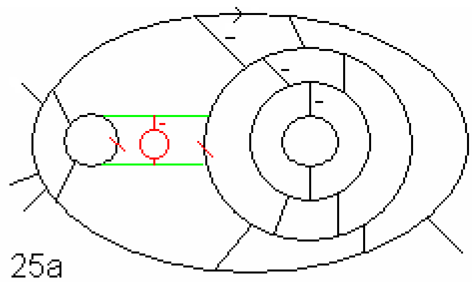


Fig. 25a

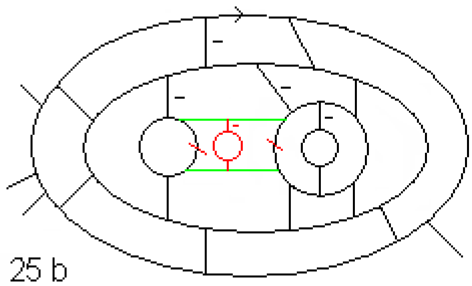


Fig. 25 b

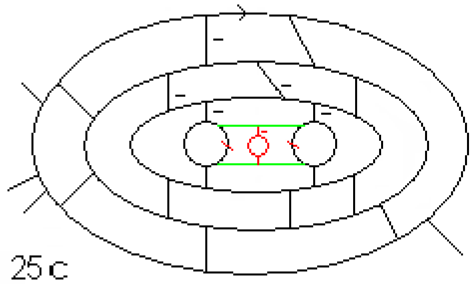


Fig. 25 c

Theorem 9:

Two concentric spoke diagrams with m respectively n circles, which are lying besides each other and have the same orientation, can by $m\ n$ (+R2b)-moves always be transformed into a single concentric spoke diagram of $(m + n)$ circles.

Proof:

From theorem 7 follows for the number N of (+R2b)-steps:

$N = [1+2+\ldots(m+n-1)] - \{[1+2+\ldots(m-1)] + [1+2+\ldots(n-1)]\} =$

$= \frac{1}{2}[(m+n-1)(m+\mathbf{n})] - \frac{1}{2}\{[(m-1)m] + [(n-1)n]\} =$

$= \frac{1}{2}[(m-1)m+nm + (n-1)n+mn] - \frac{1}{2}\{[(m-1)m] + [(n-1)n]\} =$

$= mn.$ (6.5)

□

b) Outside-transfer of a system of concentric circles (Fig. 26 and 27):

Under this is understood a (+R2b)-connexion between an enclosing circle *K* and the most outside lying circle of a system *S* of concentric circles inside, where *S* and *K* have different orientations.

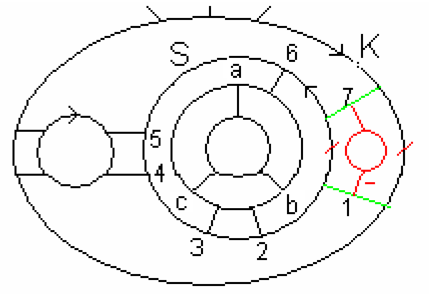


Fig. 26

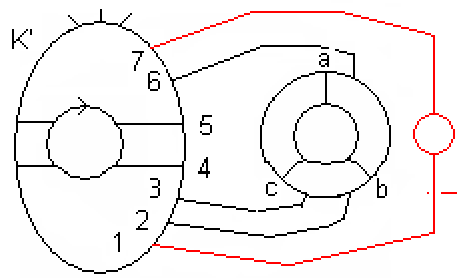


Fig. 27

Because of the different orientation of *K* and the outer circle of *S* there is possible a (+R2b)-connexion between them (s. Fig. 26). The result shows Fig. 27, where *S* (without the outer circle) is transferred outside.

7. Spoke Diagrams of Braids.

Let's consider a braid as a sequence of elementary braids with always one crossing (compare example in Fig. 28). Then we determine the spoke diagram as was shown in Fig. 1, 2 and 3 of section 1, completed with the rule for the sign (s. Fig. 29, where should be noticed, that at braids the signs of the crossings are usually denoted inverse).

Instead of whole circles we now get only vertical line pieces according to the strands of the braid. The spokes are leading horizontally between always neighboured strand-lines.

If the braid is closed (that means: each strand-line is completed to a circle) we get a concentric knot (or link) as shown in Fig. 30.

The Reidemeister moves R1, R2a and R3a do always keep a concentric knot diagram within the class of concentric diagrams, they are therefore called 'concentric R-moves'. These include the rules for equivalent changes of braid diagrams [1].

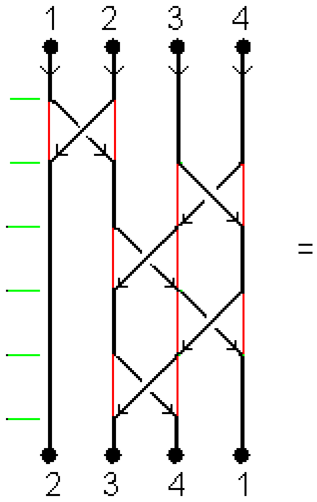


Fig. 28

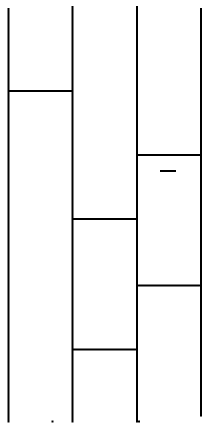


Fig. 29

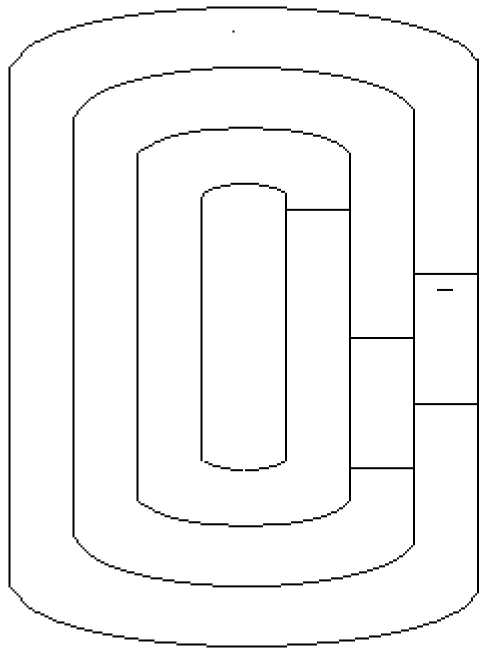


Fig. 30

Let’s consider a concentric diagram and an assigned braid to it. Then each result of moves at the braid is sufficiently valid for the concentric knot, which is won by closing the moved braid, but it is not a necessary condition.

In the Fig.31a,b,c is demonstrated for the braid of Fig.28 a sequence of braid moves.

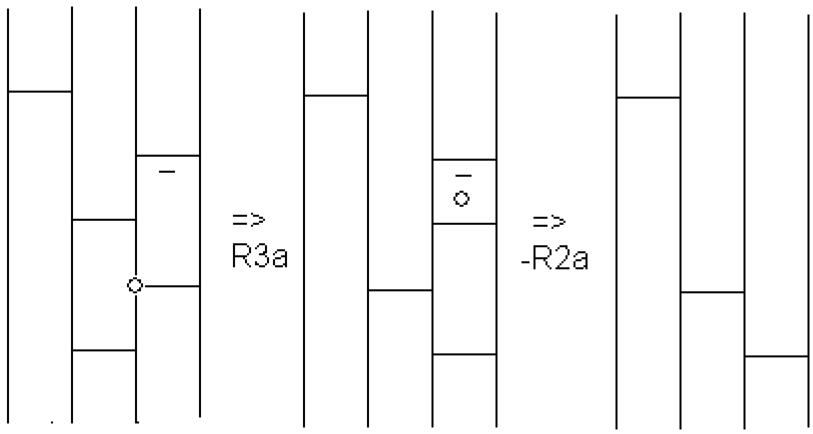


Fig.31a Fig. 31b Fig. 31c

From the algebraic word of the assigned braid of a concentric diagram can be deduced <u>a simple numerical code</u> for it [1]: just list the indices of the elementary braids and add the sign ‘-‘ for minus-crossings.

As example - for the concentric knot of Fig. 30 do we get [1,-3, 2, 3, 2]. In order to get a one-to-one representation we started before 1 in our example and generally use the order of |+/-i| and ‘+’ goes before ‘-‘. Spokes, which can obviously lie in the same radius line of the concentric diagram (as 1 and -3 in our example) are as well ordered in this way.

8. Composition of Concentric Spoke Diagrams.

A concentric spoke diagram S1 with *m* circles and another one S2 with *n* circles may be composed to one single concentric spoke diagram.

For this purpose always the most outer circle of S1 and the most outer circle of S2 are interrupted and connected by 2 lines (s. Fig. 32/a). As a result we get a surrounding circle of the concentric spoke diagrams S1' and S2' with *m* -1 and *n* -1 circles (s. Fig. 32/b).

S1' and S2' can be concentrated. Thus as a result we get again one single concentric spoke diagram. Because of theorem 9 does it have *m*+*n*-1 circles.

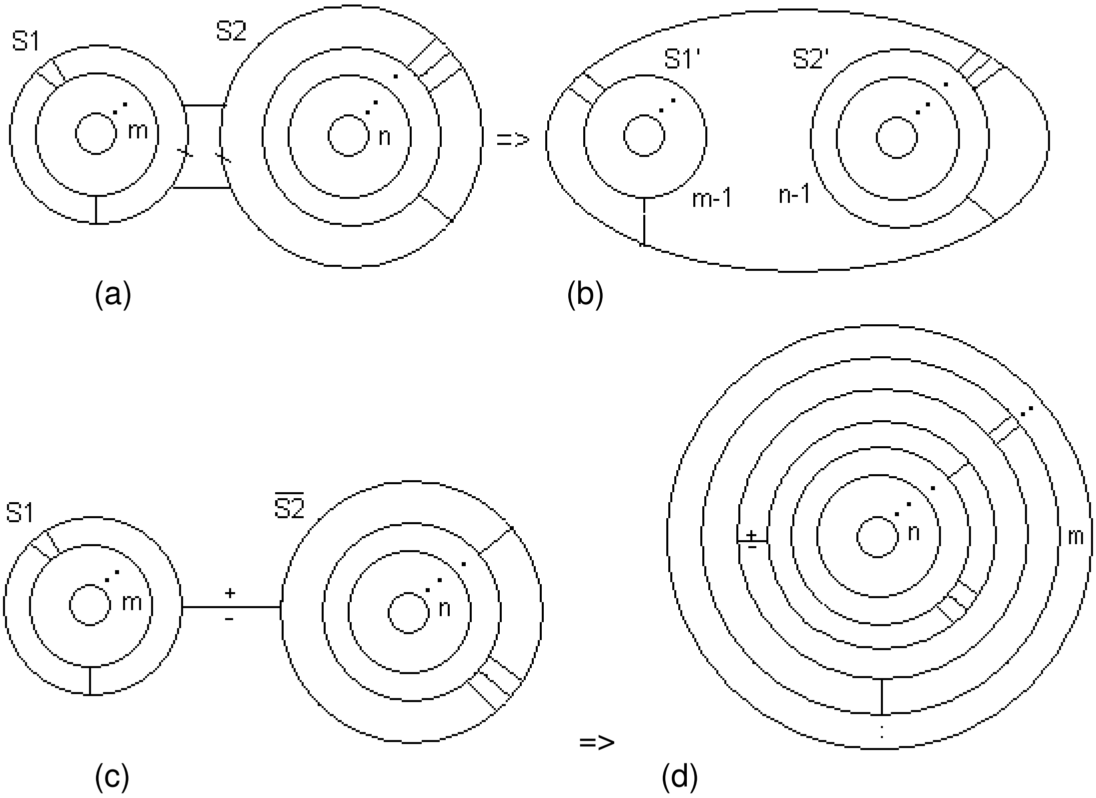


Fig. 32

On the other hand: by turning the right (or left) side from Fig. (b) by 180° around the horizontal axis, we get system S1 and system S2 connected by 1 spoke (+ or -) (s. Fig. (c)). Since the point of the connection is arbitrary, we may generally formulate

<u>Theorem 10</u>;

The composition of 2 knots (links) *A* and *B* is a 1-spoke connection between *A* and the turned knot-diagram of *B*, or a 1-spoke connection between the turned knot-diagram of *A* and *B.*

Finally – by turning one of our systems S1, S2 upside down the other we also get a centric spoke diagram (having 1 more circle). This must be equivalent to the centric diagram, which we would get above (from Fig. (b).

9. The Twin Spokes Theorem.

Definitions:

+R3 is called a Reidemeister move R3 in a concentric spoke diagram, by which the number of spokes outside is increased, *-R3* is called a move R3,, by which it is decreased.

Outside condensed is called a concentric spoke diagram, at which immediately cannot applied a (+R3)-move, as well as no (-R1)- or (-R2)-move.
Inside condensed is called a concentric spoke diagram, at which immediately cannot applied a (-R3)-move, as well as no (-R1)- or (-R2)-move.

Plus diagram is called a concentric spoke diagram of at least 1 spoke, which has only spokes of sign + (plus-spokes). *Minus diagram* is called one with only spokes of sign – (minus-spokes).

Plus twin spokes (*minus twin spokes*) is always called a pair of plus-spokes (minus-spokes), which lie in the same ring besides each other and no other spoke is put on between them in one of the neighboured rings.

Theorem 11:

An outside condensed plus diagram (minus diagram) always contains at least one pair of plus twin spokes (minus twin spokes).

Proof:

(We need only consider a plus diagram, because for a minus diagram is it corresponding the same with only changed signs).

From the outside condensed concentric diagram is considered the (from inside counted) first ring (r), which contains spokes. There must be at least 2 spokes, otherwise could be applied (-R1), contrary to the premise, that the diagram is outside condensed. Any two of the spokes, lying besides each other, are called $S_{r,1}$ and $S_{r,2}$ (s. Fig.33).

In the neighboured ring r+1 are possible the following 3 situations:
1. There is no spoke put on between $S_{r,1}$ and $S_{r,\,2}$: In this case $S_{r,1}$ and $S_{r,\,2}$ are already the intended plus twin spokes, q.e.d.
2. There is put on exactly 1 spoke: This case is contrary to the premise, because there could be applied (+R3).
3. There are at least 2 spokes; any two of them, lying besides each other are called $S_{r+1,1}$ and $S_{r+1,2}$: Then is considered the situation in the ring r+2 as was done in r+1 above, and so on until to the last ring n: Since there is no further ring n+1 respectively no further spoke in the ring n+1, the spokes $S_{n,1}$ and $S_{n,2}$ at the latest represent the twin spokes. ☐

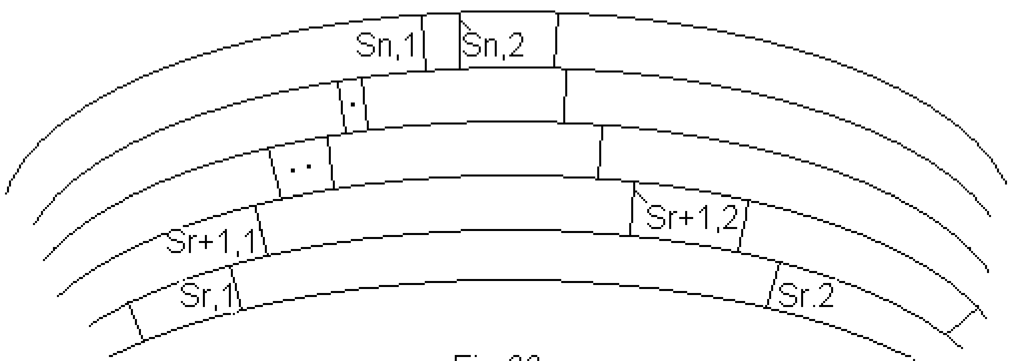


Fig.33

10. Proof of Finite Recursion at Determining Knot-Invariant Polynomials.

We consider the recurrence diagrams, which are got by the recurrence relation

$$A\,X(+) + B\,X(-) + C\,X(0) + D = 0 \qquad (10.1)$$

for purpose of determining of a knot-invariant polynomial.

As was demonstrated in section 6, the concerning knot (or oriented link) can always be represented by a *concentric* spoke diagram. If this concentric spoke diagram is used for the recurrence relation (10.1), we can immediately see, that the recurrence-reduced spoke diagrams always again form concentric spoke diagrams (with either one spoke with changed sign or one spoke omitted).

In the following is represented the logical flow chart, after which the concentric diagrams, which are always got by a recurrence step, may be advantageously treated in a computer program. The knot representation by the numerical code, which is demonstrated at the end of section 7 is specifically useful for this.*)

The '?' always either means 'is the marked execution possible?' or 'does the marked situation exist?'.

S1: <u>(-R1)</u>? Y: [execute], =>S1
    N: <u>(-R2)</u>? Y: [execute], =>S1
        N: <u>(+R3)</u>? Y: [execute], =>S1
            N: {now we have an outside condensed diagram
                or circles without spokes}
            <u>(minus twins)</u>? Y: [mark one of the 2 spokes], =>SR
                N: <u>(plus twins)</u>? Y: [mark one of the 2 spokes], =>SR
                    N: <u>(minus spoke)</u>? Y: [mark the spoke], =>SR
                        N: {no minus spoke and no plus twins means
                        because of theorem 11, that we must have
                        circles without spokes} =>SE

SR: {the diagram is now ready for a further recursion step by the recurrence relation (10.1), this is executed in relation to the marked spoke}. ….

SE: {the diagram is now ready for the final treatment: determining of the *X*-polynomial of a system of concentric circles without spokes, and determining of its coefficients.

----

*) The treatment of the numerical code of the diagram is simply the following:
<u>(-R1)?</u> means: case 1 'is there only one number 1 contained in the numerical code?' case 2 'does the largest number in the code occur only once?'
<u>(-R2)?</u> means 'is there a sequence (x,-x) or (-x, x) contained in the numerical code?'
<u>(+R3)?</u> means 'a) is there a sequence (x, x+1, x) or b1) (x, x+1,-x) or b2) (-x, x+1, x) or c1) (x,-x-1,-x) or c2) (-x,-x-1, x) or d) (-x,-x-1,-x) contained in the numerical code?'
Similar simple is always the execution of the concerning R-move at the code.

Theorem 12;

The treatment of the diagrams according to the flow chart above does after a finite number of recursion steps always lead to a system of circles without spokes.

Proof:

Let us consider the changes of the occurrence of spokes in the different recursion steps according to the flow chart above (s. Fig.34).

'Diagr.1' and 'diagr.2' always mean the 2 by the recurrence resulting diagrams.

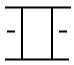
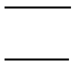
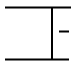
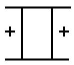
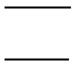
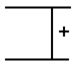
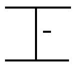
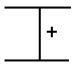
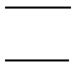
Case	diagr1,diagr.2 (after -R2)		change of number of spokes	change of number of -spokes	change of number of +spokes
1) 			-2 -1	-2 -1	0 0
2) 			-2 -1	0 0	-2 -1
3) 			0 -1	-1 -1	+1 0

Fig.34

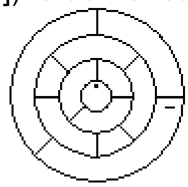
Note, that the number of minus spokes never increases. In the only case, at which the number of spokes altogether does not decrease, there decreases the number of minus spokes. This case can therefore only occur finitely often. Afterwards do we always have plus diagrams and because of theorem 11 and the flow chart above we then always get case (2) of our table, by which the number of spokes altogether is decreased in each recursion step, thus - after a finite number of these – we must always get diagrams without spokes or at most 1 spoke, which by (-R1) will also be reduced to a system of circles without any spoke.

□

APPENDIX

Centric or (with the (+R2b)-method by manual labour) centred spoke diagrams of the prime-knots and -links until 8 spokes.

As an example for the used numeric code for the representation, the knot 9_{49} (according [1]) is demonstrated:



= [1 3 2 1-3 2 2 1 3 2 2] = 9_{49} [1]

The sequence of spokes is given clockwise by the numbers of the rings, in which the spokes lie, whereat the rings are counted from the inner one to the outer. If there is a '-spoke', the minus-sign is written before the number of the ring.

If there is a succession of numbers, which do not denote neighbouring rings. the marked spokes may be laid at the same radial line (compare end of section 7).

The denotation of the prime knots respectively links follows [1], but to the denotations of links – in order to distinguish those of different orientations of their knots – are added the indices *a*, *b* (at links of 2 knots), respectively *a*, *b*, *c*, *d* (at links of 3 knots). In the case there are more -spokes than +spokes, the inverse knot (link) is represented.

Coding	Denotation [1]	Coding	Denotation [1]
1 1	2^2_{1a} 2^2_{1b}	1 1 1 2 2-1 2	-6^2_{3a}
1 1 1	-3_1	1 1 1-2 1 1-2	7^2_{4a} 7^2_{4b}
1 1 1 1	-4^2_{1a}	1 1 1-2-2 1-2	7^2_{2a}
1-2 1-2	4_1	1 1 2 1 1-2-2	6^2_{3b}
1 1 1 1 1	-5_1	1 1 2-1-1-1 2	-7^2_{7b}
1 1 2-1 2	4^2_{1b}	1 1 2 2 1-2-2	-7^2_{8a} -7^2_{8b}
1 1-2 1-2	-5^2_{1a} -5^2_{1b}	1 1-2 1 1-2-2	7^2_{5a}
1 1 1 1 1 1	-6^2_{1a}	1 1-2 1-2 1-2	-7^2_{6a} -7^2_{6b}
1 1 1-2 1-2	-6_2	1 1-3 2-1-3 2	6_1
1 1 2 1 1 2	6^3_{3a}	1 2 1 3 2 2 3	-6^3_{1d}
1 1 2-1-1 2	-6^3_{3b} -6^3_{3c}	1 3-2 1 3-2-2	-7^3_{1a}
1 1 2 2 1-2	-5_2	1 1 1 1 1 1 1 1	-8^2_{1a}
1 1-2 1 1-2	6^3_{1a} 6^3_{1b} 6^3_{1c}	1 1 1 1 1-2 1-2	-8_2
1 1-2-2 1-2	6_3	1 1 1 1-2 1 2 2	7_3
1-2 1-2 1-2	6^3_{2a} 6^3_{2b} 6^3_{2c} 6^3_{2d}	1 1 1 1-2 1-2-2	8_7
1 1 1 1 1 1 1	-7_1	1 1 1-2 1 1 1-2	8_5
1 1 1 1-2 1-2	-7^2_{1a}	1 1 1-2 1 1 2 2	-7_5
1 1 1 2 1 1 2	-7^2_{7a}	1 1 1-2 1 1-2-2	8_{10}
1 1 1 2 2 1-2	6^2_{2a} -6^2_{2b}	1 1 1-2 1-2-2-2	8_9

1 1 1 3-2 1 3-2	8^2_{5a}	1 1 3-2 3-2 1-2	8^2_{8a}
1 1 2 1 2 1 2 2	8_{19}	1-2 1-2 1-2 1-2	8_{18}
1 1-2 1 1-2 1-2	-8_{16}	1-2 1 3 2 2 3-2	-8^2_{16a}
1 1-2 1 2-1 2 2	-8_{21}	1-2 1 3-2 1 3-2	-8^2_{13a}
1 1-2-2 1 2-1 2	-8_{20}	1-2 3-2 1-2 3-2	8^2_{14a}
1 1-2-2 1-2 1-2	8_{17}	1-2 3-2 1 3-2-2	-8^2_{10a}
1 1 3-2 1 3-2-2	-8^2_{12a}	1-2 3 3-2 1-3-2	7^2_{2b}
1 1 3-2 1-3-3-2	-7^2_{1b}		

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